# CHAPTER 8

# COMBINATIONS AND PERMUTATIONS

## LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Define combinations and permutations.
2. Apply the concept of combinations to problem solving.
3. Apply the concept of principle of choice to problem solving.
4. Apply the concept of permutations to problem solving.

## INTRODUCTION

This chapter deals with concepts required for the study of probability and statistics. Statistics is a branch of science that is an outgrowth of the theory of probability. Combinations and permutations are used in both statistics and probability; and they, in turn, involve operations with factorial notation. Therefore, combinations, permutations, and factorial notation are discussed in this chapter.

## DEFINITIONS

A *combination* is defined as a possible selection of a certain number of objects taken from a group without regard to order. For instance, suppose we were to choose two letters from a group of three letters. If the group of three letters were $A$, $B$, and $C$, we could choose the letters in combinations of two as follows:

$$AB,\ AC,\ BC$$

The order in which we wrote the letters is of no concern; that is, $AB$ could be written $BA$, but we would still have only one combination of the letters $A$ and $B$.

*A permutation* is defined as a possible selection of a certain number of objects taken from a group with regard to order. The permutations of two letters from the group of three letters would be as follows:

$$AB,\ AC,\ BC,\ BA,\ CA,\ CB$$

The symbol used to indicate the foregoing combination will be ${}_3C_2$, meaning a group of three objects taken two at a time. For the previous permutation we will use ${}_3P_2$, meaning a group of three objects taken two at a time and ordered.

You will need an understanding of factorial notation before we begin a detailed discussion of combinations and permutations. We define the product of the integers $n$ through 1 as *n factorial* and use the symbol $n!$ to denote this; that is,

$$3! = 3 \cdot 2 \cdot 1$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

*EXAMPLE*: Find the value of 5!

*SOLUTION*:

$$\begin{aligned} 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 120 \end{aligned}$$

*EXAMPLE*: Find the value of

$$\frac{5!}{3!}$$

*SOLUTION*:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and

$$3! = 3 \cdot 2 \cdot 1$$

Then

$$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$$

and by simplification

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 5 \cdot 4$$

$$= 20$$

The previous example could have been solved by writing

$$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3!}{3!}$$

$$= 5 \cdot 4$$

Notice that we wrote

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and combine the factors

$$3 \cdot 2 \cdot 1$$

as

$$3!$$

so that

$$5! = 5 \cdot 4 \cdot 3!$$

*EXAMPLE*: Find the value of

$$\frac{6! - 4!}{4!}$$

*SOLUTION*:

$$6! = 6 \cdot 5 \cdot 4!$$

and

$$4! = 4! \cdot 1$$

Then

$$\frac{6! - 4!}{4!} = \frac{(6 \cdot 5 - 1)\, 4!}{4!}$$

$$= (6 \cdot 5 - 1)$$

$$= 29$$

Notice that 4! was factored from the expression

$$6! - 4!$$

Theorem. *If n and r are positive integers, with n greater than r, then*

$$n! = (n)(n-1)\cdots(r+2)(r+1)r!$$

This theorem allows us to simplify an expression as follows:

$$\begin{aligned} 5! &= 5\cdot 4! \\ &= 5\cdot 4\cdot 3! \\ &= 5\cdot 4\cdot 3\cdot 2! \\ &= 5\cdot 4\cdot 3\cdot 2\cdot 1 \end{aligned}$$

Another example is

$$\begin{aligned} (n+2)! &= (n+2)(n+1)! \\ &= (n+2)(n+1)(n!) \\ &= (n+2)(n+1)(n)\cdots 1 \end{aligned}$$

*EXAMPLE*: Simplify

$$\frac{(n+3)!}{n!}$$

*SOLUTION*:

$$(n+3)! = (n+3)(n+2)(n+1)n!$$

then

$$\begin{aligned} \frac{(n+3)!}{n!} &= \frac{(n+3)(n+2)(n+1)n!}{n!} \\ &= (n+3)(n+2)(n+1) \end{aligned}$$

**PRACTICE PROBLEMS:**

Find the value of problems 1 through 4 and simplify problems 5 and 6.

1. $6!$
2. $3!4!$
3. $\dfrac{8!}{11!}$
4. $\dfrac{5! - 3!}{3!}$
5. $\dfrac{n!}{(n - 1)!}$
6. $\dfrac{(n + 2)!}{n!}$

---

**ANSWERS:**

1. 720
2. 144
3. $\dfrac{1}{990}$
4. 19
5. $n$
6. $(n + 1)(n + 2)$

---

## COMBINATIONS

As indicated previously, a combination is the selection of a certain number of objects taken from a group of objects without regard to order. We use the symbol ${}_5C_3$ to indicate that we have five objects taken three at a time, without regard to order. Using the letters $A$, $B$, $C$, $D$, and $E$ to designate the five objects, we list the combinations as follows:

$$ABC\ ABD\ ABE\ ACD\ ACE$$

$$ADE\ BCD\ BCE\ BDE\ CDE$$

We find 10 combinations of 5 objects are taken 3 at a time. The word *combinations* implies that order is not considered.

*EXAMPLE*: Suppose we wish to know how many color combinations can be made from four different colored marbles if we use only three marbles at a time. The marbles are colored red, green, white, and yellow.

*SOLUTION*: We let the first letter in each word indicate the color; then we list the possible combinations as follows:

$$RGW \ RGY \ RWY \ GWY$$

If we rearrange the groups, for example $RGW$, to form $GWR$ or $RWG$, we still have the same color combination within each group; therefore, order is not important.

The previous examples are relatively easy to understand; but suppose we have 20 boys and wish to know how many different basketball teams we could form, one at a time, from these boys. Our listing would be quite lengthy, and we would have a difficult task to determine we had all of the possible combinations. In fact, we would have to list over 15,000 combinations. This indicates the need for a formula.

The general formula for possible combinations of $r$ objects from a group of $n$ objects is

$${}_nC_r = \frac{n(n-1)\cdot\cdot\cdot(n-r+1)}{r\cdot\cdot\cdot 3\cdot 2\cdot 1}$$

The denominator may be written as

$$r\cdot\cdot\cdot 3\cdot 2\cdot 1 = r!$$

and if we multiply both numerator and denominator by

$$(n-r)\cdot\cdot\cdot 2\cdot 1$$

which is

$$(n-r)!$$

we have

$${}_nC_r = \frac{n(n-1)\cdot\cdot\cdot(n-r+1)(n-r)\cdot\cdot\cdot 2\cdot 1}{r!(n-r)\cdot\cdot\cdot 2\cdot 1}$$

The numerator

$$n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1$$

is

$$n!$$

Therefore,

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

This formula is read: *The number of combinations of n objects taken r at a time is equal to n factorial divided by r factorial times n minus r factorial.*

*EXAMPLE*: In the previous problem where 20 boys were available, how many different basketball teams could be formed?

*SOLUTION*: If the choice of which boy played center, guard, or forward is not considered, we find the number of combinations of 20 boys taken 5 at a time by writing

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

where

$$n = 20$$

and

$$r = 5$$

Then, by substitution, we have

$$\begin{aligned} {}_nC_r = {}_{20}C_5 &= \frac{20!}{5!(20-5)!} \\ &= \frac{20!}{5!15!} \\ &= \frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15!}{5!15!} \\ &= \frac{20\cdot 19\cdot 18\cdot 17\cdot 16}{5\cdot 4\cdot 3\cdot 2\cdot 1} \\ &= \frac{1\cdot 19\cdot 3\cdot 17\cdot 16}{1} \\ &= 15{,}504 \end{aligned}$$

*EXAMPLE*: A man has , in his pocket, a silver dollar, a half-dollar, a quarter, a dime, a nickel, and a penny. If he reaches into his pocket and pulls out three coins, how many different sums may he have?

*SOLUTION*: The order is not important; therefore, the number of combinations of coins possible is

$$
\begin{aligned}
{}_6C_3 &= \frac{6!}{3!(6-3)!} \\
&= \frac{6!}{3!3!} \\
&= \frac{6\cdot 5\cdot 4\cdot 3!}{3!3!} \\
&= \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} \\
&= \frac{5\cdot 4}{1} \\
&= 20
\end{aligned}
$$

*EXAMPLE*: Find the value of

$$ {}_3C_3 $$

*SOLUTION*: We use the formula given and find that

$$
\begin{aligned}
{}_3C_3 &= \frac{3!}{3!(3-3)!} \\
&= \frac{3!}{3!0!}
\end{aligned}
$$

This seems to violate the rule, *division by zero is not allowed*, but we *define 0! to equal 1.*

Then

$$ \frac{3!}{3!0!} = \frac{3!}{3!} = 1 $$

which is obvious if we list the combinations of three things taken three at a time.

**PRACTICE PROBLEMS:**

Find the value of problems 1 through 6 and solve problems 7, 8, and 9.

1. ${}_6C_2$
2. ${}_6C_4$
3. ${}_{15}C_5$
4. ${}_7C_7$
5. $\dfrac{{}_6C_3 + {}_7C_3}{{}_{13}C_6}$
6. $\dfrac{{}_7C_3 \cdot {}_6C_3}{{}_{14}C_4}$
7. We want to paint three rooms in a house, each a different color, and we may choose from seven different colors of paint. How many color combinations are possible for the three rooms?
8. If 20 boys go out for the football team, how many different teams may be formed, one at a time?
9. Two girls and their dates go to the drive-in, and each wants a different flavored ice cream cone. The drive-in has 24 flavors of ice cream. How many combinations of flavors may be chosen among the four of them if each one selects one flavor?

---

**ANSWERS:**

1. 15
2. 15
3. 3,003
4. 1
5. $\frac{5}{156}$
6. $\frac{100}{143}$
7. 35
8. 167,960
9. 10,626

## PRINCIPLE OF CHOICE

The *principle of choice* is discussed in relation to combinations, although it is also discussed later in this chapter in relation to permutations. It is stated as follows:

*If a selection can be made in $n_1$ ways; and after this selection is made, a second selection can be made in $n_2$ ways; and after this selection is made, a third selection can be made in $n_3$ ways; and so forth for r selections, then the r selections can be made together in*

$$n_1 \cdot n_2 \cdot n_3 \cdot \cdot \cdot n_r \text{ ways}$$

*EXAMPLE*: In how many ways can a coach choose first a football team and then a basketball team from 18 boys?

*SOLUTION*: First let the coach choose a football team; that is,

$$\begin{aligned} {}_{18}C_{11} &= \frac{18!}{11!(18-11)!} \\ &= \frac{18!}{11!7!} \\ &= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11!}{11!7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= 31{,}824 \end{aligned}$$

The coach now must choose a basketball team from the remaining seven boys; that is,

$$\begin{aligned} {}_{7}C_{5} &= \frac{7!}{5!(7-5)!} \\ &= \frac{7!}{5!2!} \\ &= \frac{7 \cdot 6 \cdot 5!}{5!2!} \\ &= \frac{7 \cdot 6}{2} \\ &= 21 \end{aligned}$$

Then, together, the two teams can be chosen in

$$(31{,}824)(21) = 668{,}304 \text{ ways}$$

The same answer would be achieved if the coach chose the basketball team first and then the football team; that is,

$$
\begin{aligned}
{}_{18}C_5 \cdot {}_{13}C_{11} &= \frac{18!}{5!13!} \cdot \frac{13!}{11!2!} \\
&= (8{,}568)(78) \\
&= 668{,}304
\end{aligned}
$$

which is the same number as before.

*EXAMPLE*: A woman ordering dinner has a choice of one meat dish from four, four vegetables from seven, one salad from three, and one dessert from four. How many different menus are possible?

*SOLUTION*: The individual combinations are as follows:

| | |
|---|---|
| meat | ${}_4C_1$ |
| vegetable | ${}_7C_4$ |
| salad | ${}_3C_1$ |
| dessert | ${}_4C_1$ |

The values of these combinations are

$$
\begin{aligned}
{}_4C_1 &= \frac{4!}{1!(4-1)!} \\
&= \frac{4!}{3!} \\
&= 4
\end{aligned}
$$

and

$$
\begin{aligned}
{}_7C_4 &= \frac{7!}{4!(7-4)!} \\
&= \frac{7!}{4!3!} \\
&= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} \\
&= 35
\end{aligned}
$$

and

$$
\begin{aligned}
{}_3C_1 &= \frac{3!}{1!(3-1)!} \\
&= \frac{3!}{2!} \\
&= 3
\end{aligned}
$$

Therefore, the woman has a choice of

$$(4)(35)(3)(4) = 1{,}680$$

different menus.

---

**PRACTICE PROBLEMS:**

Solve the following problems:

1. A man has 12 different colored shirts and 20 different ties. How many shirt and tie combinations can he select to take on a trip if he takes 3 shirts and 5 ties?

2. A petty officer in charge of posting the watch has 12 men in his duty section. He must post 3 different fire watches and then post 4 aircraft guards on different aircraft. How many different assignments of men can he make?

3. If 10 third class and 14 second class petty officers are in a division that must furnish 2 second class and 6 third class petty officers for shore patrol, how many different shore patrol parties can be made?

---

**ANSWERS:**

1. 3,410,880

2. 27,720

3. 19,110

---

## PERMUTATIONS

Permutations are similar to combinations but extend the requirements of combinations by considering order.

Suppose we have two letters, $A$ and $B$, and wish to know how many arrangements of these letters can be made. Obviously the answer is two; that is,

$$AB \text{ and } BA$$

If we extend this to the three letters $A$, $B$, and $C$, we find the answer to be

$$ABC,\ ACB,\ BAC,\ BCA,\ CAB,\ CBA$$

We had three choices for the first letter; after we chose the first letter, we had only two choices for the second letter; and after the second letter, we had only one choice. This is shown in the "tree" diagram in figure 8-1. Notice that a total of six different paths lead to the ends of the "branches" of the "tree" diagram.

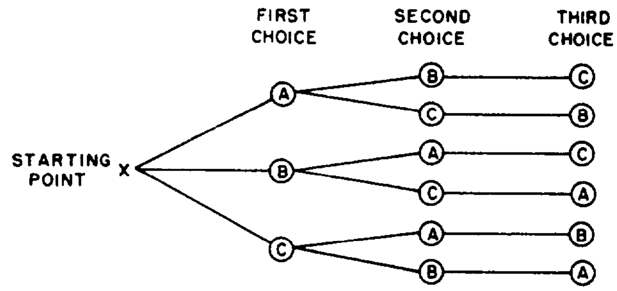


Figure 8-1.—"Tree" diagram.

If the number of objects is large, the tree would become very complicated; therefore, we approach the problem in another manner, using parentheses to show the possible choices. Suppose we were to arrange six objects in as many different orders as possible. For the first choice we have six objects:

$$(6)(\ )(\ )(\ )(\ )(\ )$$

For the second choice we have only five choices:

$$(6)(5)(\ )(\ )(\ )(\ )$$

For the third choice we have only four choices:

$$(6)(5)(4)(\ )(\ )(\ )$$

This may be continued as follows:

$$(6)(5)(4)(3)(2)(1)$$

By applying the principle of choice, we find the total possible ways of arranging the objects to be the product of the individiual choices; that is,

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and this may be written as

$$6!$$

This leads to the statement: *The number of permutations of n objects taken all together is equal to n*!.

*EXAMPLE*: How many permutations of seven different letters may be made?

*SOLUTION*: We could use a "tree" diagram, but this would become complicated. (Try it to see why.) We could use the parentheses as follows:

$$(7)(6)(5)(4)(3)(2)(1) = 5040$$

The easiest solution is to use the previous statement and write

$$_7P_7 = 7!$$

that is, the number of possible arrangements of seven objects taken seven at a time is 7!.

NOTE: Compare this with the number of combinations of seven objects taken seven at a time.

If we are faced with finding the number of permutations of seven objects taken three at a time, we use three parentheses as follows:

In the first position we have a choice of seven objects:

$$(7)(\ )(\ )$$

In the second position we have a choice of six objects:

$$(7)(6)(\ )$$

In the last position we have a choice of five objects:

$$(7)(6)(5)$$

Therefore by principle of choice, the solution is

$$7 \cdot 6 \cdot 5 = 210$$

At this point we will use our knowledge of combinations to develop a formula for the number of permutations of $n$ objects taken $r$ at a time.

Suppose we wish to find the number of permutations of five things taken three at a time. We first determine the number of groups of three as follows:

$$
\begin{aligned}
{}_5C_3 &= \frac{5!}{3!(5-3)!} \\
&= \frac{5!}{3!2!} \\
&= 10
\end{aligned}
$$

Thus, we have 10 groups of 3 objects each.

We are now asked to arrange each of these 10 groups in as many orders as possible. We know that the number of permutations of three objects taken together is 3!. We may arrange each of the 10 groups in 3! or 6 ways. The total number of possible permutations of ${}_5C_3$ then is

$$ {}_5C_3 \cdot 3! = 10 \cdot 6 $$

which can be written as

$$ {}_5C_3 \cdot 3! = {}_5P_3 $$

The corresponding general form is

$$ {}_nC_r \cdot r! = {}_nP_r $$

Knowing that

$$ {}_nC_r = \frac{n!}{r!(n-r)!} $$

then

$$
\begin{aligned}
{}_nC_r \cdot r! &= \frac{n!}{r!(n-r)!} \cdot r! \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

But

$$ {}_nC_r \cdot r! = {}_nP_r $$

therefore,

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

This formula is read: *The number of permutations of n objects taken r at a time is equal to n factorial divided by n minus r factorial.*

*EXAMPLE*: How many permutation of six objects taken two at a time can be made?

*SOLUTION*: The number of permutations of six objects taken two at a time is written

$$
\begin{aligned}
{}_6P_2 &= \frac{6!}{(6-2)!} \\
&= \frac{6!}{4!} \\
&= \frac{6 \cdot 5 \cdot 4!}{4!} \\
&= 6 \cdot 5 \\
&= 30
\end{aligned}
$$

*EXAMPLE*: In how many ways can eight people be arranged in a row?

*SOLUTION*: All eight people must be in the row; therefore, we want the number of permutations of eight people taken eight at a time, which is

$$
\begin{aligned}
{}_8P_8 &= \frac{8!}{(8-8)!} \\
&= \frac{8!}{0!}
\end{aligned}
$$

(remember that 0! was defined as equal to 1)

then

$$
\begin{aligned}
\frac{8!}{0!} &= \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} \\
&= 40{,}320
\end{aligned}
$$

Problems dealing with combinations and permutations often require the use of both formulas to solve one problem.

*EXAMPLE*: Eight first class and six second class petty officers are on the board of the 56 club. In how many ways can the members elect, from the board, a president, a vice-president, a secretary, and a treasurer if the president and secretary must be first class petty officers and the vice-president and treasurer must be second class petty officers?

*SOLUTION*: Since two of the eight first class petty officers are to fill two different offices, we write

$$
\begin{aligned}
{}_8P_2 &= \frac{8!}{(8-2)!} \\
&= \frac{8!}{6!} \\
&= 8 \cdot 7 \\
&= 56
\end{aligned}
$$

Then, two of the six second class petty officers are to fill two different offices; thus, we write

$$
\begin{aligned}
{}_6P_2 &= \frac{6!}{(6-2)!} \\
&= \frac{6!}{4!} \\
&= 6 \cdot 5 \\
&= 30
\end{aligned}
$$

The principle of choice holds in this case; therefore, the members have

$$56 \cdot 30 = 1{,}680$$

ways to select the required office holders.

*EXAMPLE*: For the preceding example, suppose we are asked the following: In how many ways can the members elect the office holders from the board if two of the office holders must be first class petty officers and two of the office holders must be second class petty officers?

*SOLUTION*: We have already determined how many ways eight things may be taken two at a time, how many ways six things may be taken two at a time, and how many ways they may be taken together; that is,

$${}_8P_2 = 56$$

and

$${}_6P_2 = 30$$

then

$${}_8P_2 \cdot {}_6P_2 = 1{,}680$$

Our problem now is to find how many ways the members can combine the four offices two at a time. Therefore, we write

$$
\begin{aligned}
{}_4C_2 &= \frac{4!}{2!(4-2)!} \\
&= \frac{4!}{2!2!} \\
&= \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 2} \\
&= 6
\end{aligned}
$$

Then, in answer to the problem, we write

$$ {}_8P_2 \cdot {}_6P_2 \cdot {}_4C_2 = 10{,}080 $$

In other words, if the members have ${}_4C_2$ ways of combining the four offices and then for every one of these ways, the members have ${}_8P_2 \cdot {}_6P_2$ ways of arranging the office holders, then they have

$$ {}_8P_2 \cdot {}_6P_2 \cdot {}_4C_2 $$

ways of electing the petty officers.

---

**PRACTICE PROBLEMS:**

Find the answers to the following:

1. ${}_6P_3$
2. ${}_4P_3$
3. ${}_7P_2 \cdot {}_5P_2$
4. In how many ways can six people be seated in a row?
5. Seven boys and nine girls are in a club. In how many ways can they elect four different officers designated by $A$, $B$, $C$, and $D$ if
   a. $A$ and $B$ must be boys and $C$ and $D$ must be girls?
   b. two of the officers must be boys and two of the officers must be girls?

**ANSWERS:**

1. 120
2. 24
3. 840
4. 720
5. a. 3,024

   b. 18,144

---

If we were asked how many different arrangements of the letters in the word *STOP* can be made, we would write

$$
\begin{aligned}
{}_4P_4 &= \frac{4!}{(4-4)!} \\
&= \frac{4!}{0!} \\
&= 24
\end{aligned}
$$

We would be correct since all letters are different. If some of the letters were the same, we would reason as given in the following problem.

*EXAMPLE*: How many different arrangements of the letters in the word *ROOM* can be made?

*SOLUTION*: We have two letters alike. If we list the possible arrangements, using subscripts to make a distinction between the *O*'s, we have

$$
\begin{array}{llll}
RO_1O_2M & O_1O_2MR & O_1MO_2R & MO_1O_2R \\
RO_2O_1M & O_2O_1MR & O_2MO_1R & MO_2O_1R \\
RO_1MO_2 & O_1O_2RM & O_1RMO_2 & MO_1RO_2 \\
RO_2MO_1 & O_2O_1RM & O_2RMO_1 & MO_2RO_1 \\
RMO_1O_2 & O_1MRO_2 & O_1RO_2M & MRO_1O_2 \\
RMO_2O_1 & O_2MRO_1 & O_2RO_1M & MRO_2O_1
\end{array}
$$

but we cannot distinguish between the $O$'s; $RO_1O_2M$ and $RO_2O_1M$ would be the same arrangement without the subscript. Only half as many arrangements are possible without the use of subscripts (a total of 12 arrangements). This leads to the statement: *The number of arrangements of n items, where there are k groups of like items of size* $r_1, r_2, \ldots r_k$, *respectively, is given by*

$$\frac{n!}{r_1!r_2! \cdots r_k!}$$

In the previous example $n$ was equal to 4 and two letters were alike; therefore, we would write

$$\frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1}$$

$$= 12$$

*EXAMPLE*: How many arrangements can be made using the letters in the word *ADAPTATION*?

*SOLUTION*: We use

$$\frac{n!}{r_1!r_2! \cdots r_k!}$$

where

$$n = 10$$

and

$$r_1 = 2 \text{ (two } T\text{'s)}$$

and

$$r_2 = 3 \text{ (three } A\text{'s)}$$

Then

$$\frac{n!}{r_1!r_2! \cdots r_k!} = \frac{10!}{2!3!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 2}{1}$$

$$= 302{,}400$$

**PRACTICE PROBLEMS:**

Find the number of possible arrangements of the letters in the following words:

1. *WRITE*

2. *STRUCTURE*

3. *HERE*

4. *MILLIAMPERE*

5. *TENNESSEE*

---

**ANSWERS:**

1. 120

2. 45,360

3. 12

4. 2,494,800

5. 3,780

---

Although the previous discussions have been associated with formulas, problems dealing with combinations and permutations may be analyzed and solved in a more meaningful way without complete reliance upon the formulas.

*EXAMPLE*: How many four-digit numbers can be formed from the digits 2, 3, 4, 5, 6, and 7

a. without repetition?

b. with repetition?

*SOLUTION*: The (a) part of the question is a straight-forward permutation problem, and we reason that we want the number of permutations of six items taken four at a time. Therefore,

$$
\begin{aligned}
{}_6P_4 &= \frac{6!}{(6-4)!} \\
&= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} \\
&= 360
\end{aligned}
$$

The (b) part of the question would become quite complicated if we tried to use the formulas; therefore, we reason as follows:

We desire a four-digit number and find we have six choices for the first position; that is, we may use any of the digits 2, 3, 4, 5, 6, or 7 in the first position:

$$(6)(\ )(\ )(\ )$$

Since we are allowed repetition of numbers, then we have six choices for the second position. In other words, any of the digits 2, 3, 4, 5, 6, or 7 can be used in the second position:

$$(6)(6)(\ )(\ )$$

Continuing this reasoning, we also have six choices each for the third and fourth positions:

$$(6)(6)(6)(6)$$

Therefore, the total number of four-digit numbers formed from the digits 2, 3, 4, 5, 6, and 7 with repetition is

$$6 \cdot 6 \cdot 6 \cdot 6 = 1{,}296$$

*EXAMPLE*: Suppose, in the previous example, we were to find how many three-digit odd numbers could be formed from the given digits without repetition.

*SOLUTION*: We would be required to start in the units column because an odd number is determined by the units column digit. Therefore, we have only three choices for the units position; that is, either 3, 5, or 7:

$$(3)(\ )(\ )$$

For the ten's position, we have only five choices, since we are not allowed repetition of numbers:

$$(3)(5)(\ )$$

Using the same reasoning of no repetition, we have only four choices for the hundred's position:

$$(3)(5)(4)$$

Therefore, we can form

$$3 \cdot 5 \cdot 4 = 60$$

three-digit odd numbers from the digits 2, 3, 4, 5, 6, and 7 without repetition.

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**PRACTICE PROBLEMS:**

Solve the following problems:

1. Using the digits 4, 5, 6, and 7, how many two-digit numbers can be formed
   a. without repetition?
   b. with repetition?
2. Using the digits 4, 5, 6, 7, 8, and 9, how many five-digit numbers can be formed
   a. without repetition?
   b. with repetition?
3. Using the digits of problem 2, how many four-digit odd numbers can be formed without repetition?

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**ANSWERS:**

1. a. 12
   b. 16
2. a. 720
   b. 7,776
3. 180

## SUMMARY

**The following are the major topics covered in this chapter:**

1. **Definitions:**

   A *combination* is defined as a possible selection of a certain number of objects taken from a group without regard to order.

   A *permutation* is defined as a possible selection of a certain number of objects taken from a group with regard to order.

   The product of the integers $n$ through 1 is defined as *n factorial*, and the symbol $n!$ is used to denote this.

2. **Factorial:** Theorem. *If n and r are positive integers, with n greater than r, then*

$$n! = (n)(n-1)\cdots(r+2)(r+1)r!$$

3. **Combination formula:**

$$_nC_r = \frac{n!}{r!(n-r)!}$$

   for the number of combinations of $n$ objects taken $r$ at a time.

4. **Principle of Choice:** If a selection can be made in $n_1$ ways; and after this selection is made, a second selection can be made in $n_2$ ways; and after this selection is made, a third selection can be made in $n_3$ ways; and so forth for $r$ selections, then the sequence of $r$ selections can be made together in $n_1 \cdot n_2 \cdot n_3 \cdots n_r$ ways.

5. **Permutation formula:**

$$_nP_r = \frac{n!}{(n-r)!}$$

   for the number of permutations of $n$ objects taken $r$ at a time.

6. **Arrangements:** The number of arrangements of $n$ items, where there are $k$ groups of like items of size $r_1, r_2, \cdots, r_k$, respectively, is given by

$$\frac{n!}{r_1!r_2! \cdots r_k!}$$

7. **Repetition**: Combinations and permutation problems, with or without repetition, may be solved for using position notation instead of formulas.

## ADDITIONAL PRACTICE PROBLEMS

1. Find the value of $\frac{6!7! - 6!5!}{5!4!}$.

2. Simplify $\frac{(n + 2)!(n - 2)! - (n + 1)!(n - 1)!}{(n)!(n - 3)!}$.

3. Find the value of $\frac{{}_7C_5 + {}_7C_6}{{}_7C_5 - {}_7C_6}$.

4. On each trip, a salesman visits 4 of the 12 cities in his territory. In how many different ways can he schedule his route?

5. From six men and five women, find the number of groups of four that can be formed consisting of two men and two women.

6. Find the value of $\frac{{}_7P_6 + {}_7P_5}{{}_7P_6 - {}_7P_5}$.

7. In how many ways can the 18 members of a boy scout troop elect a president, a vice-president, and a secretary, assuming that no member can hold more than one office?

8. How many different ways can 4 red, 3 blue, 4 yellow, and 2 green bulbs be arranged on a string of Christmas tree lights with 13 sockets?

9. How many car tags can be made if the first three positions are letters and the last three positions are numbers (Hint: Twenty-six letters and ten distinct single-digit numbers are possible)

   a. with repetition?

   b. without repetition?

## ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 1,230

2. $3(n + 1)(n - 2)$

3. 2

4. 495

5. 150

6. 3

7. 4,896

8. 900,900

9. a. 17,576,000

   b. 11,232,000